Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Sound and Vibration 281 (2005) 1175–1185
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

The use of a dynamical basis for computing the modes of a beam system with a discontinuous cross-section

Teresa Tsukazan

Instituto de Matemática, Universidade Federal do Rio Grande do Sul, P.O. Box 10673, 90.001-970 Porto Alegre, RS, Brazil

Received 9 June 2003; accepted 6 April 2004
Available online 15 December 2004

1. Introduction

In this work we determine the modes and the frequency equation of Euler–Bernoulli beams with discontinuous properties in the transversal section by using a dynamical basis which is generated by a fundamental solution of a fourth-order differential equation [1–4].

Free vibrations of stepped beams have been studied by several authors applying exact and numerical techniques. We can cite Gorman [5], Jang and Bert [6,7], Nagulseswaran [8–10], De Rosa [11–13], Vu et al. [14], Turhan [15], Korenev and Reznikov [16], Krylov [17], among others. The frequency equation and mode shapes have been formulated in terms of the classical Euler basis involving the roots of the associated characteristic polynomial of a fourth-order differential equation. The use of the dynamical basis allows the identification of factors that frequently appear in the literature, as well as writing the frequency equations and modes in a compact form.

We consider beams subject to general boundary conditions, which include non-classical conditions, such as the ones found when seeking the modes of a column partially immersed in fluid, as considered by Uŝciłowska and Kołodziej [18]. The continuity conditions of the physical properties are formulated at the position of the discontinuity. The case of an intermediate support at the discontinuity is also discussed. For the sake of clarity, in eigenanalysis we employ a matrix

E-mail address: teresa@mat.ufrgs.br (T. Tsukazan).

doi:10.1016/j.jsv.2004.04.021

approach. The same methodology can be applied to other types beams that resulting from diverse approximations as discussed by Han et al. [19].

The matrix formulation is done in such a way that the modes are obtained from the product of a boundary matrix and a basis matrix that involve values of the dynamical basis at points located on the boundary and points where the discontinuities are located. The boundary matrix carries coefficients associated with the boundary conditions and with the discontinuous properties. This allows considering beams with more transversal sections with discontinuous properties. It simply keeps the bordering lines and columns that correspond to the fixed but arbitrary boundary conditions and expands the conditions for discontinuous properties inside the boundary matrix.

In the work of Lee and Ke [20], a non-uniform beam is first approximated by a beam with a finite number of step beams. For each stepbeam a fundamental basis is introduced. The Wronskian of this basis is normalized to be the identity matrix at $x = 0$. The resulting basis is just a sort of normalization of the dynamical basis to express a solution in terms of the initial data at $x = 0$. The study of frequency equations usually involves the analysis of transcendental equations that arise by using the trigonometric–hyperbolic representation of the elements of the dynamical basis. Other approaches could be considered by writing a frequency equation in terms of a dynamical basis.

In this work we propose a simple and effective method to study the problem of the free vibration of a spring-restrained free-supported beam, which is quite different from all previous studies.

2. Modal equation for beams with discontinuous cross-sections

Let us consider a double-span Euler–Bernoulli beam with a discontinuous cross-section. A flexural movement is represented in the beam by $v_1(t,x)$ in the first segment and by $v_2(t,x)$ in the second segment. The movement is then described by the Euler–Bernoulli model [22,23] in each segment of the beam

$$\rho_1 A_1 \frac{\partial^2 v_1(t,x)}{\partial t^2} + E_1 I_1 \frac{\partial^4 v_1(t,x)}{\partial x^4} = 0, \quad 0 < x < \mu, \tag{1}$$

$$\rho_2 A_2 \frac{\partial^2 v_2(t,x)}{\partial t^2} + E_2 I_2 \frac{\partial^4 v_2(t,x)}{\partial x^4} = 0, \quad \mu < x < L, \tag{2}$$

with the usual parameter description. For free vibrations of the harmonic type whose spatial distribution amplitudes are $X_1(x)$, $X_2(x)$ in each segment, as shown in the Fig. 1, we can substitute them into Eq. (1) to obtain the spatial modal differential equations

$$X_1^{(\mathrm{iv})}(x) - a_1\rho_1 A_1 X_1(x) = 0, \quad X_2^{(\mathrm{iv})}(x) - a_2\rho_2 A_2 X_2(x) = 0, \tag{3,4}$$

where

$$a_i = \frac{\omega^2}{E_i I_i}, \quad i = 1, 2.$$

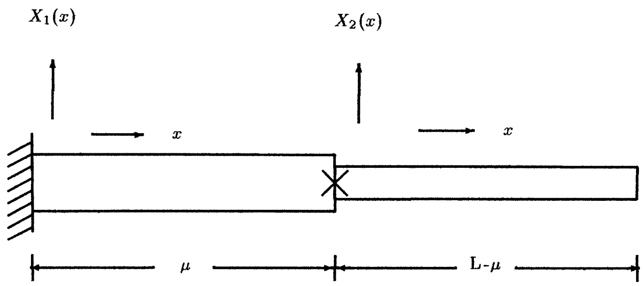

Fig. 1. A bi-segmented beam with a discontinuity in the cross-section and an intermediate device.

The mode is then given by

$$X(x) = \begin{cases} X_1(x), & 0 \leqslant x \leqslant \mu, \\ X_2(x), & \mu \leqslant x \leqslant L. \end{cases} \tag{5}$$

For the case of beams with one discontinuity, the solution for each segment can be conveniently written as

$$X_1(x) = d_{11}\phi_1 + d_{21}\phi_2 + d_{31}\phi_3 + d_{41}\phi_4 = \mathbf{\Psi}_1(x)\mathbf{d}_1,$$

$$X_2(x) = d_{12}\psi_1 + d_{22}\psi_2 + d_{32}\psi_3 + d_{42}\psi_4 = \mathbf{\Psi}_2(x)\mathbf{d}_2,$$

where $\mathbf{\Psi}_1(x) = [\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)]$ is a solution basis of Eq. (3), $\mathbf{d}_1$ is the column vector with components $d_{11}, d_{21}, d_{31}, d_{41}$ employed for describing the mode in the first segment, $\mathbf{\Psi}_2(x) = [\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)]$ is a solution basis of Eq. (4), and $\mathbf{d}_2$ is the column vector with components $d_{12}, d_{22}, d_{32}, d_{42}$ employed for describing the mode in the second segment. Generic boundary conditions of classical or non-classical nature can be written as

$$\begin{aligned} &A_{11}X_1(0) + B_{11}X_1'(0) + C_{11}X_1''(0) + D_{11}X_1'''(0) = 0, \\ &A_{12}X_1(0) + B_{12}X_1'(0) + C_{12}X_1''(0) + D_{12}X_1'''(0) = 0, \\ &A_{21}X_2(L) + B_{21}X_2'(L) + C_{21}X_2''(L) + D_{21}X_2'''(L) = 0, \\ &A_{22}X_2(L) + B_{22}X_2'(L) + C_{22}X_2''(L) + D_{22}X_2'''(L) = 0. \end{aligned} \tag{6}$$

The continuity conditions for the displacement, the inertia moment, the bending moment and the shear force at the discontinuity point μ of the transversal section, including an intermediate device, can be written in general as

$$\begin{aligned} &E_{11}X_1(\mu) + F_{11}X_1'(\mu) + G_{11}X_1''(\mu) + H_{11}X_1'''(\mu) \\ &\quad = E_{12}X_2(\mu) + F_{12}X_2'(\mu) + G_{12}X_2''(\mu) + H_{12}X_2'''(\mu), \\ &E_{21}X_1(\mu) + F_{21}X_1'(\mu) + G_{21}X_1''(\mu) + H_{21}X_1'''(\mu) \\ &\quad = E_{22}X_2(\mu) + F_{22}X_2'(\mu) + G_{22}X_2''(\mu) + H_{22}X_2'''(\mu). \end{aligned} \tag{7}$$

The substitution of Eq. (5) with the above description into the boundary and continuity conditions leads to the matrix system

$$\mathscr{U}\mathbf{c} = \mathbf{0}, \tag{8}$$

where

$$\mathscr{U} = \mathscr{B}\mathbf{\Phi}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix}.$$

Here, the matrix $\mathscr{B}$ carries the coefficients associated with the boundary and continuity conditions (6) and (7). It is given by

$$\mathscr{B} = \begin{bmatrix} A_{11} & B_{11} & C_{11} & D_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{12} & B_{12} & C_{12} & D_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{11} & F_{11} & G_{11} & H_{11} & -E_{12} & -F_{12} & -G_{12} & -H_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{21} & F_{21} & G_{21} & H_{21} & -E_{22} & -F_{22} & -G_{22} & -H_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{31} & F_{31} & G_{31} & H_{31} & -E_{32} & -F_{32} & -G_{32} & -H_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{41} & F_{41} & G_{41} & H_{41} & -E_{42} & -F_{42} & -G_{42} & -H_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{21} & B_{22} & C_{21} & D_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22} & B_{21} & C_{22} & D_{22} \end{bmatrix}. \tag{9}$$

The matrix $\mathbf{\Phi}$ carries the values of the solution basis at the ends and the conditions at the discontinuity point, that is

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) & 0 & 0 & 0 & 0 \\ \phi_1'(0) & \phi_2'(0) & \phi_3'(0) & \phi_4'(0) & 0 & 0 & 0 & 0 \\ \phi_1''(0) & \phi_2''(0) & \phi_3''(0) & \phi_4''(0) & 0 & 0 & 0 & 0 \\ \phi_1'''(0) & \phi_2'''(0) & \phi_3'''(0) & \phi_4'''(0) & 0 & 0 & 0 & 0 \\ \phi_1(\mu) & \phi_2(\mu) & \phi_3(\mu) & \phi_4(\mu) & 0 & 0 & 0 & 0 \\ \phi_1'(\mu) & \phi_2'(\mu) & \phi_3'(\mu) & \phi_4'(\mu) & 0 & 0 & 0 & 0 \\ \phi_1''(\mu) & \phi_2''(\mu) & \phi_3''(\mu) & \phi_4''(\mu) & 0 & 0 & 0 & 0 \\ \phi_1'''(\mu) & \phi_2'''(\mu) & \phi_3'''(\mu) & \phi_4'''(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_1(\mu) & \psi_2(\mu) & \psi_3(\mu) & \psi_4(\mu) \\ 0 & 0 & 0 & 0 & \psi_1'(\mu) & \psi_2'(\mu) & \psi_3'(\mu) & \psi_4'(\mu) \\ 0 & 0 & 0 & 0 & \psi_1''(\mu) & \psi_2''(\mu) & \psi_3''(\mu) & \psi_4''(\mu) \\ 0 & 0 & 0 & 0 & \psi_1'''(\mu) & \psi_2'''(\mu) & \psi_3'''(\mu) & \psi_4'''(\mu) \\ 0 & 0 & 0 & 0 & \psi_1(L) & \psi_2(L) & \psi_3(L) & \psi_4(L) \\ 0 & 0 & 0 & 0 & \psi_1'(L) & \psi_2'(L) & \psi_3(L) & \psi_4'(L) \\ 0 & 0 & 0 & 0 & \psi_1''(L) & \psi_2''(L) & \psi_3(L) & \psi_4''(L) \\ 0 & 0 & 0 & 0 & \psi_1'''(L) & \psi_2'''(L) & \psi_3'''(L) & \psi_4'''(L) \end{bmatrix}. \tag{10}$$

Non-zero solutions of Eq. (8) are obtained for frequency values that satisfy the characteristic equation

$$\det(\mathscr{U}) = 0. \tag{11}$$

3. The dynamical basis

To solve the modal equation (8), it is necessary to introduce a basis suitable for determining matrix (10). From the many mathematical bases available for the fourth-order equation

$$X^{(\mathrm{iv})}(x) - \varepsilon^4 X(x) = 0, \quad \varepsilon^4 = a\rho A = \frac{\omega^4 \rho A}{E^2 I^2}, \tag{12}$$

it is convenient to choose one that makes Eq. (10) as sparse as possible. This is accomplished by choosing the *dynamical* or *fundamental basis* [1,3] which is generated by the solution $h(x)$ of the initial value problem

$$\begin{aligned} &h^{(\mathrm{iv})}(x) - \varepsilon^4 h(x) = 0, \\ &h(0) = 0, \quad h'(0) = 0, \quad h''(0) = 0, \quad h'''(0) = 1, \end{aligned} \tag{13}$$

and it first three derivatives $h'(x)$, $h''(x)$, $h'''(x)$. In terms of the traditional spectral basis, constructed using the roots of the associated characteristic polynomial $P(s) = s^4 - \varepsilon^4$, that is,

$$\mathbf{\Psi} = [\sin(\varepsilon x), \cos(\varepsilon x), \sinh(\varepsilon x), \cosh(\varepsilon x)],$$

we have that the fundamental solution $h(x)$ has the following representation with respect to the spectral Euler basis:

$$h(x) = \frac{\sinh(\varepsilon x) - \sinh(\varepsilon x)}{2\varepsilon^3}. \tag{14}$$

As mentioned in the introduction, the dynamical basis appears in the mathematical literature as well as factors or as normalized basis in several works already mentioned.

We consider in the first segment the dynamical basis

$$\mathbf{\Psi}_1 = [h(x,\varepsilon), h'(x,\varepsilon), h''(x,\varepsilon), h'''(x,\varepsilon)]$$

and the translated dynamical basis

$$\mathbf{\Psi}_2 = [h(x-\mu,\varepsilon), h'(x-\mu,\varepsilon), h''(x-\mu,\varepsilon), h'''(x-\mu,\varepsilon)]$$

for the second segment. Then we choose

$$\phi_j(x) = h^{(j-1)}(x, \varepsilon_1), \quad \psi_j(x) = h^{(j-1)}(x-\mu, \varepsilon_2), \quad j = 1,2,3,4.$$

The fundamental response $h(x,\varepsilon)$, has the same shape for each segment, but depends on different values that the following parameters defined in Ref. [6]:

$$\varepsilon_1 = \sqrt[4]{a_1^2 \rho_1 A_1}, \quad \varepsilon_2 = \theta\varepsilon_1, \quad \theta = \frac{\varphi}{\alpha}, \tag{15}$$

$$\varphi = \sqrt[4]{\frac{\rho_2 A_2}{\rho_1 A_1}}, \quad \alpha = \sqrt[4]{\frac{E_2 I_2}{E_1 I_1}}, \tag{16}$$

take in each segment of the beam. The frequency will be then given by

$$\omega = \varepsilon_1^2 \sqrt{\frac{E_1 I_1}{\rho_1 A_1}}. \tag{17}$$

Thus

$$\phi_1(x) = h(x, \varepsilon_1) = \frac{\sinh\,(\varepsilon_1 x) - \sin\,(\varepsilon_1 x)}{2\varepsilon_1^3},$$
$$\psi_1(x) = h(x-\mu, \varepsilon_2) = \frac{\sinh\,(\varepsilon_2(x-\mu)) - \sin\,(\varepsilon_2(x-\mu))}{2\varepsilon_2^3}.$$

Replacing the values at $x = 0$, using the initial values of $h(x)$ and differentiating Eq. (13) for higher-order derivatives, we have that matrix (10) with basis values becomes more sparse, that is

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_1(\mu) & \phi_2(\mu) & \phi_3(\mu) & \phi_4(\mu) & 0 & 0 & 0 & 0 \\ \phi_1'(\mu) & \phi_2'(\mu) & \phi_3'(\mu) & \phi_4'(\mu) & 0 & 0 & 0 & 0 \\ \phi_1''(\mu) & \phi_2''(\mu) & \phi_3''(\mu) & \phi_4''(\mu) & 0 & 0 & 0 & 0 \\ \phi_1'''(\mu) & \phi_2'''(\mu) & \phi_3'''(\mu) & \phi_4'''(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_1(L) & \psi_2(L) & \psi_3(L) & \psi_4(L) \\ 0 & 0 & 0 & 0 & \psi_1'(L) & \psi_2'(L) & \psi_3'(L) & \psi_4'(L) \\ 0 & 0 & 0 & 0 & \psi_1''(L) & \psi_2''(L) & \psi_3''(L) & \psi_4''(L) \\ 0 & 0 & 0 & 0 & \psi_1'''(L) & \psi_2'''(L) & \psi_3'''(L) & \psi_4'''(L) \end{bmatrix}. \tag{18}$$

Let us set

$$\phi_{ij}(x) = h^{(i+j-2)}(x, \varepsilon_1), \quad \psi_{ij}(x) = h^{(i+j-2)}(x-\mu, \varepsilon_2). \tag{19}$$

Then the modal equation $\mathcal{U}\mathbf{c} = \mathbf{0}$ has

$$U = \begin{bmatrix} D_{11} & C_{11} & B_{11} & A_{11} & 0 & 0 & 0 & 0 \\ D_{12} & C_{12} & B_{12} & A_{12} & 0 & 0 & 0 & 0 \\ U_{11} & U_{12} & U_{13} & U_{14} & -H_{12} & -G_{12} & -F_{12} & -E_{12} \\ U_{21} & U_{22} & U_{23} & U_{24} & -H_{22} & -G_{22} & -F_{22} & -E_{22} \\ U_{31} & U_{32} & U_{33} & U_{34} & -H_{32} & -G_{32} & -F_{32} & -E_{32} \\ U_{41} & U_{42} & U_{43} & U_{44} & -H_{42} & -G_{42} & -F_{42} & -E_{42} \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & V_{13} & V_{14} \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & V_{23} & V_{24} \end{bmatrix}, \tag{20}$$

where

$$U_{ij} = E_{i1}\phi_{1j}(\mu) + F_{i1}\phi_{2j}(\mu) + G_{i1}\phi_{3j}(\mu) + H_{i1}\phi_{4j}(\mu), \quad i,j = 1,2,3,4,$$
$$V_{ij} = A_{2i}\psi_{1j}(L) + B_{2i}\psi_{2j}(L) + C_{2i}\psi_{3j}(L) + D_{2i}\psi_{4j}(L), \quad i = 1,2; \quad j = 1,2,3,4. \tag{21}$$

4. Modes for discontinuous beams without intermediate devices

For a beam with discontinuous cross-section without any device at the discontinuity point μ of the transverse section, the continuity conditions are

$$X_1(\mu) = X_2(\mu),\ X_1'(\mu) = X_2'(\mu), \tag{22}$$

$$X_1''(\mu) = \alpha^4 X_2''(\mu),\ X_1'''(\mu) = \alpha^4 X_2'''(\mu), \tag{23}$$

where

$$\alpha = \sqrt[4]{\frac{E_2 I_2}{E_1 I_1}}. \tag{24}$$

For this kind of conditions, the matrix $\mathscr{B}$ becomes

$$\mathscr{B} = \begin{bmatrix} A_{11} & B_{11} & C_{11} & D_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{12} & B_{12} & C_{12} & D_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\alpha^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\alpha^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{21} & B_{21} & C_{21} & D_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22} & B_{22} & C_{22} & D_{22} \end{bmatrix}. \tag{25}$$

It follows that the modal matrix is of the type

$$\mathscr{U} = \begin{bmatrix} D_{11} & C_{11} & B_{11} & A_{11} & 0 & 0 & 0 & 0 \\ D_{12} & C_{12} & B_{12} & A_{12} & 0 & 0 & 0 & 0 \\ \phi_{11}(\mu) & \phi_{12}(\mu) & \phi_{13}(\mu) & \phi_{14}(\mu) & 0 & 0 & 0 & -1 \\ \phi_{21}(\mu) & \phi_{22}(\mu) & \phi_{23}(\mu) & \phi_{24}(\mu) & 0 & 0 & -1 & 0 \\ \phi_{31}(\mu) & \phi_{32}(\mu) & \phi_{33}(\mu) & \phi_{34}(\mu) & 0 & -\alpha^4 & 0 & 0 \\ \phi_{41}(\mu) & \phi_{42}(\mu) & \phi_{43}(\mu) & \phi_{44}(\mu) & -\alpha^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & V_{13} & V_{14} \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & V_{23} & V_{24} \end{bmatrix}, \tag{26}$$

where the components V_{ij} are determined from Eq. (21).

5. Discontinuous beams with an intermediate support

For a beam with an intermediate support at the discontinuity $x = \mu$, the matrix $\mathscr{B}$ becomes more sparse since the displacement condition at the discontinuity point μ is now zero and the shear force is not considered. We have

$$\mathscr{B} = \begin{bmatrix} A_{11} & B_{11} & C_{11} & D_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{12} & B_{12} & C_{12} & D_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\alpha^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{21} & B_{21} & C_{21} & D_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22} & B_{22} & C_{22} & D_{22} \end{bmatrix}.$$

It follows that the modal matrix is of the type

$$\mathscr{U} = \begin{bmatrix} D_{11} & C_{11} & B_{11} & A_{11} & 0 & 0 & 0 & 0 \\ D_{12} & C_{12} & B_{12} & A_{12} & 0 & 0 & 0 & 0 \\ \phi_{11}(\mu) & \phi_{12}(\mu) & \phi_{13}(\mu) & \phi_{14}(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \phi_{21}(\mu) & \phi_{22}(\mu) & \phi_{23}(\mu) & \phi_{24}(\mu) & 0 & 0 & -1 & 0 \\ \phi_{31}(\mu) & \phi_{32}(\mu) & \phi_{33}(\mu) & \phi_{34}(\mu) & 0 & -\alpha^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & V_{13} & V_{14} \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & V_{23} & V_{24} \end{bmatrix}, \tag{27}$$

where the components V_{ij} are computed from Eq. (21) by substituting the coefficients in Eq. (5) corresponding to the continuity conditions. We observe that the presence of an intermediate support simplifies the matrix $\mathscr{U}$ and the modal analysis becomes simpler.

6. A spring-restrained free-supported double span beam with an intermediate support

For a free-supported double span beam with an intermediate support and restrained with a rotational spring of elasticity constant $k = \kappa EI/L$ at the right end, we have

$$\mathscr{U} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h(\mu,\varepsilon_1) & h'(\mu,\varepsilon_1) & h''(\mu,\varepsilon_1) & h'''(\mu,\varepsilon_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ h'(\mu,\varepsilon_1) & h''(\mu,\varepsilon_1) & h'''(\mu,\varepsilon_1) & h^{(\mathrm{iv})}(\mu,\varepsilon_1) & 0 & 0 & -1 & 0 \\ h''(\mu,\varepsilon_1) & h'''(\mu,\varepsilon_1) & h^{(\mathrm{iv})}(\mu,\varepsilon_1) & h^{(\mathrm{v})}(\mu,\varepsilon_1) & 0 & -\alpha^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & h(\gamma,\varepsilon_2) & h'(\gamma,\varepsilon_2) & h''(\gamma,\varepsilon_2) & h'''(\gamma,\varepsilon_2) \\ 0 & 0 & 0 & 0 & U_{85} & U_{86} & U_{87} & U_{88} \end{bmatrix}, \quad (28)$$

where

$$U_{85} = \kappa h'(\gamma,\varepsilon_2) + Lh''(\gamma,\varepsilon_2), \quad U_{86} = \kappa h''(\gamma,\varepsilon_2) + Lh'''(\gamma,\varepsilon_2),$$
$$U_{87} = \kappa h'''(\gamma,\varepsilon_2) + Lh^{(\mathrm{iv})}(\gamma,\varepsilon_2), \quad U_{88} = \kappa h^{(\mathrm{iv})}(\gamma,\varepsilon_2) + Lh^{(\mathrm{v})}(\gamma,\varepsilon_2).$$

From the modal equation, it follows that $d_{13} = d_{14} = d_{24} = 0$. Setting $d_{11} = 1$, we get

$$h''(\mu,\varepsilon_1) + h'''(\mu,\varepsilon_1)d_{14} = 0, \quad h'''(\mu,\varepsilon_1) + h^{(\mathrm{iv})}(\mu,\varepsilon_1)d_{14} - d_{23} = 0,$$
$$h^{(\mathrm{iv})}(\mu,\varepsilon_1) + h^{(\mathrm{v})}(\mu,\varepsilon_1)d_{12} - \alpha^4 d_{22} = 0,$$
$$h(\gamma,\varepsilon_2)d_{21} + h'(\gamma,\varepsilon_2)d_{22} + h''(\gamma,\varepsilon_2)d_{23} = 0,$$

$$(\kappa h'(\gamma,\varepsilon_2) + Lh''(\gamma,\varepsilon_2))d_{21} + (\kappa h''(\gamma,\varepsilon_2) + Lh'''(\gamma,\varepsilon_2))d_{22} + (\kappa h'''(\gamma,\varepsilon_2) + Lh''(\gamma,\varepsilon_2))d_{23} = 0.$$

The modes are then given by

$$X(x) = \begin{cases} h(x,\varepsilon_1) + \sigma_{12}h'(x,\varepsilon_1), & 0 \leqslant x \leqslant \mu, \\ \sigma_{21}h(x,\varepsilon_2) + \sigma_{22}h'(x,\varepsilon_2) + \sigma_{23}h''(x,\varepsilon_2), & \mu \leqslant x \leqslant L, \end{cases}$$

where

$$\sigma_{12} = d_{12}, \quad \sigma_{21} = d_{21}, \quad \sigma_{22} = D_{22}, \quad \sigma_{23} = d_{23},$$

$$d_{23} = h'''(\mu,\varepsilon_1) + \varepsilon_1^4 h(\mu,\varepsilon_1)d_{12}, \quad d_{22} = \varepsilon_1^4[h(\mu,\varepsilon_1) + h'(\mu,\varepsilon_1)]d_{12},$$

$$d_{21} = \frac{\varepsilon_1^4 h'(\gamma,\varepsilon_2)h'(\mu,\varepsilon_1) + \varepsilon_1^4 h''(\gamma,\varepsilon_2)h(\mu,\varepsilon_1)[\alpha^4 d_{12} + 1] + \alpha^4 h''(\gamma,\varepsilon_2)h'''(\mu,\varepsilon_1)}{\alpha^4 h(\gamma,\varepsilon_2)},$$

$$d_{12} = \frac{h''(\mu,\varepsilon_1)}{h'''(\mu,\varepsilon_1)}.$$

As before, the substitution of these coefficients into the last equation of the system $\mathscr{U}\mathbf{c} = 0$ leads to the characteristic equation

$$\Delta = \det(\mathscr{U}) = h''(\gamma,\varepsilon_2)d_{21} + h'''(\gamma,\varepsilon_2)d_{22} + h^{(\mathrm{iv})}(\gamma,\varepsilon_2)d_{23} = 0.$$

7. Concluding remarks

The frequency equation and the modal system of Euler–Bernoulli beams with discontinuous cross-section have been formulated in terms of the dynamical basis generated by the fundamental solution $h(x)$ of a fourth-order differential equation. This allowed us to determine the modes in a systematic manner in terms of $h(x)$ and its derivatives. This methodology can be applied to other kinds of beams that result from diverse approximations such as Rayleigh, shear beams or Timoshenko beams as discussed in Ref. [19]. In this case, Eqs. (18)–(21) are of the same form. Only shape (14) of the generating solution $h(x)$ of the dynamical basis changes for each model. The use of the fundamental solution allows us to reproduce by limit operations the known results for continuous cross-section beams. Also, we should observe that the two methodologies proposed by Low [21] can be unified by choosing the dynamical basis, since $h(x)$ is actually the Laplace inverse transform of the transfer function of the fourth-order differential equation.

Acknowledgements

The author would like to thank Prof. Julio R. Claeyssen for noting the value of using a dynamical basis for computing the vibration modes of a beam system.

References

[1] J.R. Claeyssen, R.A. Soder, A dynamical basis for computing modes of Euler–Bernoulli and Timoshenko beams, *Journal of Sound and Vibration* 259 (4) (2003) 986–990.
[2] K.S. Miller, *Linear Differential Equations in the Real Domain*, Norton, New York, 1963.
[3] J.R. Claeyssen, G.C. Suazo, C. Jung, A direct approach to second-order matrix non-classical vibrating equations, *Applied Numerical Mathematics* 39 (1999) 65–78.
[4] J.R. Claeyssen, T. Tsukazan, Dynamic solutions of linear matrix differential equations, *Quarterly of Applied Mathematics* XLVIII (1) (1990).
[5] D. Gorman, *Free Vibration Analysis of Beams and Shafts*, Wiley, New York, 1975.
[6] S.K. Jang, C.W. Bert, Free vibrations of stepped beams: exact and numerical solutions, *Journal of Sound and Vibration* 130 (2) (1989) 342–346.
[7] S.K. Jang, C.W. Bert, Free vibrations of stepped beams: higher mode frequencies and effects of steps on frequency, *Journal of Sound and Vibration* 132 (1) (1989) 164–168.
[8] S. Naguleswaran, Natural frequencies, sensitivity and mode shape details on Euler–Bernoulli beam with one step change in cross-section and with ends on classical supports, *Journal of Sound and Vibration* 252 (4) (2002) 751–767.
[9] S. Naguleswaran, Vibration of an Euler–Bernoulli beam on elastic end supports and with up to three step changes in cross-section, *International Journal of Mechanical Sciences* 44 (2002) 2541–2555.
[10] S. Naguleswara, Lateral vibration of a uniform Euler–Bernoulli beam carrying a particle at an intermediate point, *Journal of Sound and Vibration* 43 (1999) 2737–2752.
[11] M.A. De Rosa, M.J. Maurizi, The influence of concentrated masses and pasternak soil in the free vibrations of Euler beams-exact solution, *Journal of Sound and Vibration* 212 (4) (1998) 573–581.
[12] M.A. De Rosa, P.M. Belles, M.J. Maurizi, Free vibrations of stepped beams with intermediate elastic supports, *Journal of Sound and Vibration* 181 (3) (1995) 905–910.
[13] M.A. De Rosa, Free vibrations of stepped beams with elastic ends, *Journal of Sound and Vibration* 173 (4) (1994) 563–567.

[14] H.V. Vu, A.M. Ordoñez, B.H. Karnopp, Vibration of a double beam system, *Journal of Sound and Vibration* 229 (4) (2000) 807–822.
[15] Ö. Turhan, On the eigencharacteristics of longitudinally vibrating rods with a cross-section discontinuity, *Journal of Sound and Vibration* 248 (1) (2001) 167–177.
[16] B.G. Korenev, L.M. Reznikov, *Dynamic Vibration Absorbers*, Wiley, New York, 1993.
[17] A.N. Krylov, *Ship Vibration*, Collected works, Vol. 10, ANSSSR, Moscow, Lenningrad, 1948.
[18] A. Uŝciłowska, J.A. Kołodziej, Free vibrations of immersed column carrying a tip mass, *Journal of Sound and Vibration* 216 (1) (1998) 147–157.
[19] S.M. Han, H. Benaroya, T. Wei, Dynamics of transversally vibrating beams using four engineering theories, *Journal of Sound and Vibration* 225 (5) (1999) 935–988.
[20] S.Y. Lee, H.Y. Ke, Free vibrations of non-uniform beams resting on non-uniform elastic foundation with general elastic end restraints, *Computers and Structures* 14 (3) (1990) 421–429.
[21] K.H. Low, On the method to derive frequency equations of beams carrying multiple masses, *International Journal of Mechanical Sciences* 43 (2001) 871–881.
[22] J. Ginsberg, *Mechanical and Structural Vibrations*, Wiley, New York, 2002.
[23] S.K. Clark, *Dynamics of Continuous Elements*, Prentice-Hall, Englewood Cliffs, NJ, 1972.